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ON $(n,1)$-ABSORBING δ-PRIMARY IDEALS

ABDELHAQ EL KHALFI, HINDA HAMRAOUI, AND HICHAM LAARABI

Abstract. Let R be a commutative ring with nonzero identity. Let $\mathcal{I}(R)$ be the set of all ideals of R and let $\delta : \mathcal{I}(R) \longrightarrow \mathcal{I}(R)$ be a function. Then δ is called an expansion function of ideals of R if whenever L, I, J are ideals of R with $J \subseteq I$, we have $L \subseteq \delta(L)$ and $\delta(J) \subseteq \delta(I)$. Let δ be an expansion function of ideals of R. In this paper, we introduce and investigate a new class of ideals that is closely related to the class of δ-primary ideals. A proper ideal I of R is said to be an $(n,1)$-absorbing δ-primary ideal if whenever nonunit elements $a_1, a_2, ..., a_{n+1} \in R$ and $a_1a_2...a_{n+1} \in I$, then $a_1a_2...a_n \in I$ or $a_{n+1} \in \delta(I)$. Moreover, we give some basic properties of this class of ideals and we study the $(n,1)$-absorbing δ-primary ideals of the localization of rings, the direct product of rings and the trivial ring extensions.

1. Introduction

Throughout this paper, all rings are assumed to be commutative with $1 \neq 0$. If R is a ring, an ideal I of R is said to be proper if $I \neq R$, we denotes $\sqrt{I}$ the *radical* of an ideal I of R. A commutative ring R with exactly one maximal ideal is called a *local* ring. Let also $\mathrm{Spec}(R)$ denotes the set of all prime ideals of R.

Since prime ideals have an important role of ideal theory in the commutative rings, has been widely studied by various authors. Among the many recent generalizations of the notion of prime ideals in the literature, we find the following, due to Badawi [3]. A proper ideal I of a ring R is said to be a 2-absorbing ideal if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. In this case $\sqrt{I} = P$ is a prime ideal with $P^2 \subseteq I$ or $\sqrt{I} = P_1 \cap P_2$ where P_1, P_2 are incomparable prime ideals with $\sqrt{I}^2 \subseteq I$, cf. [3, Theorem 2.4]. Recently, Badawi and Yetkin [6] consider a new class of ideals called the class of 1-absorbing primary ideals. A proper ideal I of a ring R is called a 1-absorbing primary ideal of R if whenever nonunit elements $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $c \in \sqrt{I}$. In [17], A. Yassine et. al introduced the concept of 1-absorbing prime ideals which is a generalization of prime ideals. A proper ideal I of a ring R is a 1-absorbing prime ideal if whenever we take nonunit elements $a, b, c \in R$ with $abc \in I$, then $ab \in I$ or $c \in I$. In

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this case $\sqrt{I} = P$ is a prime ideal, cf. [17, Theorem 2.3]. If R is a ring in which exists a 1-absorbing prime ideal that is not prime, then R is a local ring, that is a ring with one maximal ideal.

Let $\mathcal{I}(R)$ be the set of all ideals of a ring R. Zhao [18] introduced the concept of expansion of ideals of R. We recall from [18] that a function $\delta : \mathcal{I}(R) \longrightarrow \mathcal{I}(R)$ is called an expansion function of ideals of R if whenever L, I, J are ideals of R with $J \subseteq I$, we have $L \subseteq \delta(L)$ and $\delta(J) \subseteq \delta(I)$. Note that there are explanatory examples of expansion functions included in [18, Example 1.2] and [5, Example 1]. In addition, recall from [18] that a proper ideal I of R is said to be a δ-primary ideal of R if whenever $a, b \in R$ with $ab \in I$, we have $a \in I$ or $b \in \delta(I)$, where δ is an expansion function of ideals of R. In [10], A. El Khalfi et Al. introduced the concept of 1-absorbing δ-primary ideals which is a generalization of 1-absorbing primary ideals. A proper ideal I of a ring R is a 1-absorbing δ-primary ideal if whenever we take nonunit elements $a, b, c \in R$ with $abc \in I$, then $ab \in I$ or $c \in \delta(I)$. Also, recall from [7] that a proper ideal I of R is called a δ-semiprimary ideal of R if $ab \in I$ implies $a \in \delta(I)$ or $b \in \delta(I)$. In this paper, we introduce and investigate a new concept of ideals that is closely related to the class of δ-primary ideals. Let n be a nonzero positive integer. A proper ideal I of R is said to be an $(n, 1)$-absorbing δ-primary ideal if whenever nonunit elements $a_1, a_2, ..., a_{n+1} \in R$ and $a_1a_2...a_{n+1} \in I$, then $a_1a_2...a_n \in I$ or $a_{n+1} \in \delta(I)$.

Among many results in this paper, we show that if a ring R admits an $(n+1, 1)$-absorbing δ-primary ideal of R that is not an $(n, 1)$-absorbing δ-primary ideal, then R is a local ring. Moreover, we prove that if R is a chained ring with maximal ideal M and I is an $(n, 1)$-absorbing δ-primary ideal of R that is not an $(n-1, 1)$-absorbing δ-semiprimary, then $I = M^k$ for some $1 \leq k \leq n$. Finally, we give an idea about $(n, 1)$-absorbing δ-primary ideals of the localization of rings, the direct product of rings and the trivial ring extensions.

2. Main Results

We start this section by the following definition.

Definition 2.1. *Let n be a nonzero positive integer.*

(1) A proper ideal I of a ring R is called an $(n, 1)$-absorbing δ-primary ideal if whenever $a_1a_2...a_{n+1} \in I$ for some nonunit elements $a_1, a_2, ..., a_{n+1} \in R$, then $a_1a_2...a_n \in I$ or $a_{n+1} \in \delta(I)$.

(2) A proper ideal I of a ring R is called an $(n, 1)$-absorbing δ-semiprimary ideal if whenever $a_1a_2...a_{n+1} \in I$ for some nonunit elements $a_1, a_2, ..., a_{n+1} \in R$, then $a_1a_2...a_n \in \delta(I)$ or $a_{n+1} \in \delta(I)$

.

Remark 2.2. *Let n be a nonzero positive integer and R be a ring, I a proper ideal of R and δ be an expansion function of $\mathcal{I}(\mathcal{R})$.*

(1) *If I is an $(n,1)$-absorbing δ-primary ideal. Then I is an $(n+1,1)$-absorbing δ-primary ideal.*
(2) *Every δ-primary ideal is a $(k,1)$-absorbing δ-primary ideal, for each positive integer $k \geq 2$.*
(3) *Every $(n,1)$-absorbing δ-primary ideal is an $(n,1)$-absorbing δ-semiprimary ideal.*
(4) *Let γ be an expansion function of $\mathcal{I}(\mathcal{R})$ such that $\delta(I) \subseteq \gamma(I)$. If I is an $(n,1)$-absorbing δ-primary ideal of R, then I is an $(n,1)$-absorbing γ-primary ideal of R.*

Next, we give an example of an $(n,1)$-absorbing δ-primary ideal that is not a 1-absorbing prime ideal.

Example 2.3. *Let $R := K[[X_1, X_2, ..., X_n]]$ be a ring of formal power series where K is a field for some positive integer n. Consider the expansion function $\delta : \mathcal{I}(\mathcal{R}) \longrightarrow \mathcal{I}(\mathcal{R})$ defined by $\delta(I) = I + M$ where $M = (X_1, X_2, ..., X_n)$. Consider the ideal $I = (X_1X_2...X_n)$ of R. Thus, I is not a 1-absorbing prime ideal of R since $X_1X_2...X_n \in I$ but neither $X_1X_2 \in I$ nor $X_3...X_n \in I$. Now, let $x_1, ...x_n, x_{n+1}$ be nonunit elements of R such that $x_1...x_nx_{n+1} \in I$. Clearly I is an $(n,1)$-absorbing δ-primary because $x_{n+1} \in \delta(I) = M$.*

Proposition 2.4. *Let n be a nonzero positive integer and I be a $(n,1)$-absorbing δ-primary ideal of R and $\delta(I)$ be a radical ideal, that is, $\sqrt{\delta(I)} = \delta(I)$. Then I is a δ-semiprimary ideal of R.*

Proof. Suppose that $ab \in I$ for some $a, b \in R$. Then we may assume that a, b are nonunits. Thus $a^n b \in I$ implies that $a^n \in I$ or $b \in \delta(I)$. Then we have $a \in \sqrt{I} \subseteq \sqrt{\delta(I)} = \delta(I)$ or $b \in \delta(I)$. Hence, I is a δ-semiprimary ideal of R. □

In the next result, we show that if a ring R admits an $(n+1,1)$-absorbing δ-primary ideal that is not an $(n,1)$-δ-primary ideal, then R is a local ring.

Theorem 2.5. *Let n be a nonzero positive integer and δ be an ideal expansion. Suppose that a ring R admits an $(n+1,1)$-absorbing δ-primary ideal that is not an $(n,1)$-absorbing δ-primary ideal. Then R is a local ring. Moreover, if R is not local then a proper ideal I of R is an $(n,1)$-absorbing δ-primary ideal if and only if I is a δ-primary ideal.*

Proof. Assume that I is an $(n+1,1)$-absorbing δ-primary ideal that is not an $(n,1)$-absorbing δ-primary ideal of R. Hence there exists a nonunit elements $a_1, a_2..., a_{n+1} \in R$ such that $a_1a_2...a_{n+1} \in I$ and $a_1...a_n \notin I$ and $a_{n+1} \notin \delta(I)$. Let d be a nonunit element of R. As $da_1a_2...a_{n+1} \in I$, I is an $(n+1,1)$-absorbing δ-primary ideal of R and $a_{n+1} \notin \delta(I)$, we conclude that $da_1a_2...a_n \in I$. Let c be a unit element of R. Suppose that $d + c$ is a nonunit element of R. Since $(d + c)a_1a_2...a_{n+1} \in I$, I is an $(n+1,1)$-absorbing δ-primary ideal of R and $a_{n+1} \notin \delta(I)$, we get that $(d+c)a_1a_2...a_n = da_1a_2...a_n + ca_1a_2...a_n \in I$. Since $da_1a_2...a_n \in I$, we conclude that $a_1a_2...a_n \in I$, which gives a contradiction. Hence, $d + c$ is a unit

element of R. Now, the result follows from [6, Lemma 1].
Moreover, if R is not local, we conclude by induction on n and Remark 2.2(2) that a proper ideal I of R is an $(n,1)$-absorbing δ-primary ideal if and only if I is a δ-primary ideal.

□

Next, we give a method to construct $(n,1)$-absorbing δ-primary ideals that are not $(n-1,1)$-absorbing δ-primary ideals.

Theorem 2.6. *Let n be a nonzero positive integer such that $n \geq 2$ and R be a local ring with maximal ideal M and δ be an ideal expansion. Let x be a nonzero prime element of R such that $\delta(xM^{n-1}) \subsetneq M$. If $x \in \delta(xM^{n-1})$, then xM^{n-1} is an $(n,1)$-absorbing δ-primary ideal of R. If $xM^{n-1} \neq xM^{n-2}$, then xM^{n-1} is not an $(n-1,1)$-absorbing δ-primary ideal of R.*

Proof. First, we will show that xM^{n-1} is an $(n,1)$-absorbing δ-primary ideal of R. Assume that $a_1a_2...a_{n+1} \in xM^{n-1}$ for some nonunit elements $a_1, a_2, ..., a_{n+1} \in R$. If $a_1a_2...a_n \notin xM^{n-1}$, then for all $1 \leq i \leq n$, $a_i \notin xR$, so $a_1a_2...a_n \notin xR$ because x is a prime element of R. Moreover, the fact that $a_1a_2...a_{n+1} \in xR$ and $a_1a_2...a_n \notin xR$ implies that $a_{n+1} \in xR \subseteq \delta(xM^{n-1})$. Now, we prove that xM^{n-1} is not an $(n-1,1)$-absorbing δ-primary ideal of R. By hypothesis, $xM^{n-1} \neq xM^{n-2}$, there exists some nonunit elements $a_2, ..., a_{n-1} \in R$ such that $xa_2...a_{n-1} \notin xM^{n-1}$. Let $m \in M \backslash \delta(xM^{n-1})$, then $xa_2...a_{n-1}m \in xM^{n-1}$, therfore xM^{n-1} is not an $(n-1,1)$-absorbing δ-primary ideal of R. □

Theorem 2.7. *Let n be a nonzero positive integer and I be an $(n,1)$-absorbing δ-primary ideal of a ring R where δ is an ideal expansion and let $d \in R \setminus I$ be a nonunit element of R. Then $(I : d) = \{x \in R \mid dx \in I\}$ is an $(n-1,1)$-absorbing δ-primary ideal of R. In particular, for every proper ideal J of R with $J \nsubseteq I$, $(I : J)$ is an $(n-1,1)$-absorbing δ-primary ideal of R.*

Proof. Suppose that $a_1a_2...a_n \in (I : d)$ for some elements $a_1, a_2, ..., a_n \in R$. Without loss of generality, we may assume that $a_1, ..., a_n$ are nonunit elements of R. Suppose that $a_1a_2...a_{n-1} \notin (I : d)$. Since $da_1a_2...a_n \in I$ and I is an $(n,1)$-absorbing δ-primary ideal of R, we conclude that $a_n \in \delta(I)$. So, $a_n \in \delta((I : d))$ and this completes the proof. The rest is similar. □

Proposition 2.8. *Let n be a nonzero positive integer and R be a ring, δ an ideal expansion and I be a proper ideal of R. If I is an $(n,1)$-absorbing δ-primary ideal of R, then either I is an $(n-1,1)$-absorbing δ-semiprimary ideal of R or R is local, say with maximal ideal M, such that $M^n \subseteq I$.*

Proof. If R is not local, then Theorem 2.5 and Remark 2.2(3) implies that I is $(n-1,1)$-absorbing δ-sermiprimary ideal of R. Now, assume that R is local with maximal ideal M such that I is not an $(n-1,1)$-absorbing δ-semiprimary ideal of R, then there existe some nonunite elements of R,

$a_1, a_2..., a_n \in M$ such that $a_1a_2...a_n \in I$ and $a_1a_2...a_{n-1} \notin \delta(I)$ and $a_n \notin \delta(I)$. To prove that $M^n \subseteq I$, it suffices to show that $x_1x_2...x_n \in I$ for all $x_1, x_2, ..., x_n \in M$. Let $x_1, x_2, ..., x_n \in M$. Then $x_1x_2...x_na_1a_2...a_n \in I$. Since $a_n \notin \delta(I)$ and I is an $(n,1)$-absorbing δ-primary ideal, we conclude that $x_1x_2...x_na_1a_2...a_{n-1} \in I$. Again, since $a_1a_2...a_{n-1} \notin \delta(I)$ and I is an $(n,1)$-absorbing δ-primary ideal, we have that $x_1x_2...x_n \in I$. □

Recall that a ring R is a chained ring if the set of all ideals of R is linearly ordered by inclusion. We next determine the $(n,1)$-absorbing δ-primary ideals of a chained ring.

Theorem 2.9. *Let R be a chained ring with maximal ideal M, δ an ideal expansion and I be an $(n,1)$-absorbing δ-primary for some positive integer $n \geq 2$. If I is not an $(n-1,1)$-absorbing δ-semiprimary. Then $I = M^k$ for some positive integer k, $2 \leq k \leq n$.*

Proof. By Proposition 2.8 R is local with maximal M such that $M^n \subseteq I$. Let $k = min\{i/M^i \subseteq I\}$, we show that $I = M^k$. Suppose that $M^k \subsetneq I$. Thus there is $a \in M^{k-1} \backslash I$ and $b \in I \backslash M^k$. Since R is a chained ring, we conclude that $b \in aR$. Hence $b = ar$ for some $r \in M$. Thus $b \in M^k$, a contradiction. Hence $I = M^k$.

□

Proposition 2.10. *Let n be a nonzero positive integer and $\{J_i \mid i \in D\}$ be a directed set of $(n,1)$-absorbing δ-primary ideals of R, where δ is an ideal expansion. Then the ideal $J = \cup_{i \in D} J_i$ is an $(n,1)$-absorbing δ-primary ideal of R.*

Proof. Let $a_1a_2...a_{n+1} \in J$ for some nonunits $a_1, a_2, ..., a_{n+1} \in R$, then $a_1a_2...a_{n+1} \in J_i$ for some $i \in D$. Since J_i is an $(n,1)$-absorbing δ-primary ideal of R, $a_1a_2...a_n \in J_i$ or $a_{n+1} \in \delta(J_i) \subseteq \delta(J)$. Hence, J is an $(n,1)$-absorbing δ-primary ideal of R. □

Proposition 2.11. *Let n be a nonzero positive integer and I be an $(n,1)$-absorbing δ-primary ideal of R such that $\sqrt{\delta(I)} \subseteq \delta(\sqrt{I})$, where δ is an ideal expansion. Then, $\sqrt{I}$ is a δ-primary ideal of R.*

Proof. Let $xy \in \sqrt{I}$ such that $x \notin \sqrt{I}$. Thus, there exists a positive integer p such that $x^py^p \in I$. So, $x^{n-1}x^py^p \in I$. Since I is an $(n,1)$-absorbing δ-primary ideal of R and $x^{n-1+p} \notin I$, we conclude that $y^p \in \delta(I)$. That implies $y \in \sqrt{\delta(I)} \subseteq \delta(\sqrt{I})$ and so $\sqrt{I}$ is a δ-primary ideal of R. □

We say that a proper ideal I of a ring R is an $(n,1)$-absorbing prime ideal if whenever $a_1a_2...a_{n+1} \in I$ for some nonunit elements $a_1, a_2, ..., a_{n+1} \in R$, then $a_1a_2...a_n \in I$ or $a_{n+1} \in I$.

Proposition 2.12. *Let n be a nonzero positive integer and I be a proper ideal of a ring R and δ be an ideal expansion such that $\delta(\delta(I)) = \delta(I)$. Then the following statements are satisfied.*

(1) If I is an $(n,1)$-absorbing δ-primary ideal and $a_1, a_2, ..., a_n$ are nonunit elements with $a_1a_2...a_n \notin I$, then $\delta(I : a_1a_2...a_n) = \delta(I)$.

(2) $\delta(I)$ is an $(n,1)$-absorbing δ-primary ideal of R if and only if $\delta(I)$ is an $(n,1)$-absorbing prime ideal of R

Proof. (1) Let I be an $(n,1)$-absorbing δ-primary ideal and $a_1a_2...a_n \notin I$. Note that $I \subseteq (I : a_1a_2...a_n)$ and so $\delta(I) \subseteq \delta(I : a_1a_2...a_n)$. Let $c \in (I : a_1a_2...a_n)$. Then $c \in \delta(I)$ since $a_1a_2...a_nc \in I$ and $a_1a_2...a_n \notin I$. Thus $(I : a_1a_2...a_n) \subseteq \delta(I)$. We get $\delta(I : a_1a_2...a_n) \subseteq \delta(\delta(I)) = \delta(I)$. Hence we conclude the equality.

(2) Let $a_1a_2...a_{n+1} \in \delta(I)$ for some nonunit elements $a_1, a_2, ..., a_{n+1} \in R$. Hence $a_1a_2...a_n \in \delta(I)$ or $a_{n+1} \in \delta(\delta(I)) = \delta(I)$. Thus $\delta(I)$ is an $(n,1)$-absorbing prime ideal of R. □

Proposition 2.13. *Let n be a nonzero positive integer and R be a ring, I a proper ideal of R and δ be an ideal expansion. Then I is an $(n,1)$-absorbing δ-primary ideal if and only if whenever $I_1I_2...I_{n+1} \subseteq I$ for some proper ideals $I_1, I_2, ..., I_{n+1}$ of R, then $I_1...I_n \subseteq I$ or $I_{n+1} \subseteq \delta(I)$.*

Proof. It suffices to prove the "if" assertion. Suppose that I is an $(n,1)$-absorbing δ-primary ideal and let I_1, I_2, ..., I_{n+1} be proper ideals of R such that $I_1I_2...I_{n+1} \subseteq I$ and $I_{n+1} \nsubseteq \delta(I)$. Thus $a_1a_2...a_{n+1} \in I$ for every $a_i \in I_i$, with $1 \leq i \leq n$ and $a_{n+1} \in I_{n+1} \setminus \delta(I)$. Since I is an$(n,1)$-absorbing δ-primary ideal, we then have $I_1I_2...I_n \subseteq I$, as desired. □

Recall from [18] that an ideal expansion δ is said to be intersection preserving if $\delta(I_1 \cap I_2 \cap ... \cap I_s) = \delta(I_1) \cap \delta(I_2) \cap ... \cap \delta(I_s)$ for any ideals $I_1, ..., I_s$ of R.

Proposition 2.14. *Let $(n,s) \in \mathbb{N}^{*2}$ and δ be an intersection preserving ideal expansion. If $I_1, I_2, ..., I_s$ are $(n,1)$-absorbing δ-primary ideals of R, and $\delta(I_i) = P$ for some ideal P of R and all $i \in \{1,2,...,s\}$, then $I_1 \cap I_2 \cap ... \cap I_s$ is an $(n,1)$-absorbing δ-primary ideal of R.*

Proof. Let $a_1a_2...a_{n+1} \in J = I_1 \cap I_2 \cap ... \cap I_s$ such that $a_1a_2...a_n \notin J$. Let $i \in \{1,2,...,s\}$ such that $a_1a_2...a_n \notin I_i$. Since $a_1a_2...a_{n+1} \in I_i$ and I_i is an $(n,1)$-absorbing δ-primary ideal, we conclude that $a_{n+1} \in \delta(I_i) = \delta(J)$. Therefore, J is an $(n,1)$-absorbing δ-primary ideal of R. □

Proposition 2.15. *Let n be a nonzero positive integer and R be a ring and δ be an expansion function of $\mathcal{I}(R)$. Then the following statements are equivalent.*

(1) *Every proper principal ideal is an $(n,1)$-absorbing δ-primary ideal of R.*
(2) *Every proper ideal is an $(n,1)$-absorbing δ-primary ideal of R.*

Proof. Assume that (1) holds and let I be a proper ideal of R. Let $a_1, a_2, ..., a_{n+1}$ be nonunit elements of R such that $a_1a_2...a_{n+1} \in I$. Hence $a_1a_2...a_{n+1} \in a_1a_2...a_{n+1}R$ which implies that $a_1a_2...a_n \in a_1a_2...a_{n+1}R \subseteq I$ or $a_{n+1} \in$

$\delta(a_1a_2...a_{n+1}R) \subseteq \delta(I)$. Therefore I is an $(n,1)$-absorbing δ-primary ideal of R. The converse is clear. □

An expansion function δ of $\mathcal{I}(R)$ is said to satisfy *condition* $(*)$ if $\delta(I) \neq R$ for each proper ideal I of R. Note that the identity function and the radical operation are examples of expansion functions satisfying condition $(*)$.

Theorem 2.16. *Let n be a nonzero positive integer and R be a ring and δ an expansion function of $\mathcal{I}(R)$ satisfying condition $(*)$ and $\delta(Jac(R)) = Jac(R)$. Suppose that $\delta(xI) = x\delta(I)$ for every proper ideal I of R and every $x \in R$. The following statements are equivalent.*

(i) Every proper principal ideal is an $(n,1)$-absorbing δ-primary ideal of R.

(ii) Every proper ideal is an $(n,1)$-absorbing δ-primary ideal of R.

(iii) R is local with $Jac(R)^n = (0)$.

Proof. $(i) \Leftrightarrow (ii)$ Follows from Proposition 2.15.

$(ii) \Rightarrow (iii)$ We will show that R is a local ring. Choose maximal ideals M_1, M_2 of R. Now, put $I = M_1 \cap M_2$. Since $M_1^n M_2 \subseteq I$ and I is an $(n,1)$-absorbing δ-primary ideal, we have either $M_1^n \subseteq I \subseteq M_2$ or $M_2 \subseteq \delta(I) \subseteq \delta(M_1)$.

Case 1: Suppose that $M_1^n \subseteq M_2$. Since M_2 is prime, clearly we have $M_1 \subseteq M_2$ which implies that $M_1 = M_2$.

Case 2: Suppose that $M_2 \subseteq \delta(M_1)$. Since δ satisfies condition $(*)$, $\delta(M_1)$ is proper. As $M_1 \subseteq \delta(M_1)$ and M_1 is a maximal ideal, we have $M_1 = \delta(M_1)$. Then we get $M_2 \subseteq M_1$, which implies that $M_1 = M_2$. Therefore, R is a local ring.

Now, we will prove that $Jac(R)^n = (0)$. We may assume that $Jac(R) \neq (0)$. So, let $x_1, x_2, ..., x_n \in Jac(R)$ and we choose $z \in Jac(R) \setminus (0)$. Since $x_1...x_nz \in (x_1...x_nz)$ and $(x_1...x_nz)$ is an $(n,1)$-absorbing δ-primary, we get $x_1...x_n \in (x_1...x_nz)$ or $z \in \delta((x_1...x_nz)) = z\delta((x_1...x_n))$. First, assume that $x_1...x_n \in (x_1...x_nz)$. Then there exists $r \in R$ such that $x_1...x_n = rx_1...x_nz$, which implies that $x_1...x_n(1-rz) = 0$. Since $1-rz$ is unit, we have $x_1...x_n = 0$ which completes the proof. Now, assume that $x_1...x_n \notin (x_1...x_nz)$, that is, $z \in \delta((x_1...x_nz)) = z\delta((x_1...x_n))$. Then there exists $a \in \delta((x_1...x_n)) \subseteq Jac(R)$ such that $z = za$. This implies that $z(1-a) = 0$ so that $z = 0$, a contradiction. Therefore, $Jac(R)^n = (0)$.

$(iii) \Rightarrow (i)$ Suppose that R is a local ring with $Jac(R)^n = (0)$. Let I be a proper ideal of R and $a_1a_2...a_{n+1} \in I$ for some nonunits $a_1, a_2, ..., a_{n+1} \in R$. Then $a_1, a_2, ..., a_{n+1} \in Jac(R)$ since R is local. As $Jac(R)^n = (0)$, we have $a_1a_2...a_n = 0 \in I$. Therefore, I is an $(n,1)$-absorbing δ-primary ideal of R. □

It can be easily seen that, in Theorem 2.16, (iii) always implies (i) without any assumption on δ. But we give some examples showing that the converse is not true if we drop the aforementioned assumptions on δ.

Example 2.17. *Let $R = \mathbb{Z}_{p^n}$, where p is a prime number, n is a nonzero positive integer and $\delta(I) = R$ for every proper ideal I of R. Note that δ does not satisfy condition $(*)$ and note that every ideal I of R is $(n,1)$-absorbing δ-primary. Thus $Jac(R)^n \neq (0)$, while R is a local ring.*

Example 2.18. *Let k be a field and consider the formal power series ring $R = k[[X]]$. Then R is a local ring with unique maximal ideal $m = (X)$. Define expansion function δ as $\delta(I) = \sqrt{I}$ for every ideal I of R. Then it is easy to see that every ideal of R is an $(n,1)$-absorbing δ-primary ideal, for every nonzero positive integer n. Also, it is clear that δ satisfies condition $(*)$ and $\delta(Jac(R)) = Jac(R)$ but not satisfy the condition $\delta(xI) = x\delta(I)$. Furthermore, $Jac(R)^n \neq (0)$. Thus Theorem 2.16 fails without assumption $\delta(xI) = x\delta(I)$.*

Let $f : R \to S$ be a ring homomorphism and δ, γ expansion functions of $\mathcal{I}(R)$ and $\mathcal{I}(S)$ respectively. Recall from [5] that f is called a $\delta\gamma$-homomorphism if $\delta(f^{-1}(I)) = f^{-1}(\gamma(I))$ for each ideal I of S. Also note that if f is a $\delta\gamma$-epimorphism and I is an ideal of R containing $ker(f)$, then $\gamma(f(I)) = f(\delta(I))$.

Theorem 2.19. *Let n be a nonzero positive integer and $f : R \to S$ be a ring $\delta\gamma$-homomorphism where δ, γ are expansion functions of $\mathcal{I}(R)$ and $\mathcal{I}(S)$ respectively. Suppose that f(a) is nonunit in S for every nonunit element a in R. Then the following statements hold.*

1. *If J is an $(n,1)$-absorbing γ-primary ideal of S, then $f^{-1}(J)$ is an $(n,1)$-absorbing δ-primary ideal of R.*
2. *If f is an epimorphism and I is a proper ideal of R containing* $\ker(f)$, *then I is an $(n,1)$-absorbing δ-primary ideal of R if and only if $f(I)$ is an $(n,1)$-absorbing γ-primary ideal of S.*

Proof. (1) Assume that $a_1a_2...a_{n+1} \in f^{-1}(J)$, for some nonunit elements $a_1, a_2, ..., a_{n+1} \in R$. Then $f(a_1)f(a_2)...f(a_{n+1}) \in J$. Thus $f(a_1)f(a_2)...f(a_n) \in J$ or $f(a_{n+1}) \in \gamma(J)$, which implies that $a_1...a_n \in f^{-1}(J)$ or $a_{n+1} \in f^{-1}(\gamma(J)) = \delta(f^{-1}(J))$. Therefore, $f^{-1}(J)$ is an $(n,1)$-absorbing δ-primary ideal of R.
(2) Suppose that $f(I)$ is an $(n,1)$-absorbing γ-primary ideal of S. Since $I = f^{-1}(f(I))$, we conclude that I is an $(n,1)$-absorbing δ-primary ideal of R by (1). Conversely, let $x_1, x_2, ..., x_{n+1}$ be nonunit elements of S with $x_1x_2...x_{n+1} \in f(I)$. Then there exist $a_1, a_2, ..., a_{n+1} \in R$ such that $x_i = f(a_i)$, with $1 \leq i \leq n+1$ and $f(a_1a_2...a_{n+1}) = x_1x_2...x_{n+1} \in f(I)$. Since $\ker(f) \subseteq I$, we then have $a_1a_2...a_{n+1} \in I$. Since I is an $(n,1)$-absorbing δ-primary ideal of R and $a_1a_2...a_{n+1} \in I$, we conclude that $a_1a_2...a_n \in I$ or $a_{n+1} \in \delta(I)$ which gives that $x_1x_2...x_n \in f(I)$ or $x_{n+1} \in f(\delta(I)) = \gamma(f(I))$. Thus $f(I)$ is an $(n,1)$-absorbing δ-primary ideal of S. □

Let δ be an expansion function of $\mathcal{I(R)}$ and I an ideal of R. Then the function $\bar{\delta}:\frac{R}{I}\longrightarrow\frac{R}{I}$ defined by $\bar{\delta}(\frac{J}{I})=\frac{\delta(J)}{I}$ for all ideals $I\subseteq J$, becomes an expansion function of $\frac{R}{I}$. Then, we have the following result.

Corollary 2.20. *Let n be a nonzero positive integer and R be a ring, δ an expansion function of $\mathcal{I(R)}$ and $I\subseteq J$ be proper ideals of R. Assume that $a+I$ is a nonunit element of $\frac{R}{I}$ for every nonunit element $a\in R$. Then J is an $(n,1)$-absorbing δ-primary ideal of R if and only if $\frac{J}{I}$ is an $(n,1)$-absorbing $\bar{\delta}$-primary ideal of $\frac{R}{I}$.*

Proposition 2.21. *Let n be a nonzero positive integer and S be a multiplicatively closed subset of a ring R and δ_S an expansion function of $\mathcal{I}(S^{-1}R)$ such that $\delta_S(S^{-1}I)=S^{-1}(\delta(I))$ for each ideal I of R. If I is abn $(n,1)$-absorbing δ-primary ideal of R such that $I\cap S=\varnothing$, then $S^{-1}I$ is an $(n,1)$-absorbing δ_S-primary ideal of $S^{-1}R$.*

Proof. Let I be an $(n,1)$-absorbing δ-primary ideal of R such that $I\cap S=\varnothing$ and $\frac{a_1}{s_1}\frac{a_2}{s_2}...\frac{a_{n+1}}{s_{n+1}}\in S^{-1}I$ for some nonunit elements $a_1,a_2,...,a_{n+1}\in R$ and $s_1,s_2,...,s_{n+1}\in S$ such that $\frac{a_1}{s_1}...\frac{a_n}{s_n}\notin S^{-1}I$. Then $xa_1a_2...a_{n+1}\in I$ for some $x\in S$. Since I is an $(n,1)$-absorbing δ-primary and $xa_1...a_n\notin I$, we conclude that $a_{n+1}\in\delta(I)$. Thus $\frac{a_{n+1}}{s_{n+1}}\in S^{-1}(\delta(I))=\delta_S(S^{-1}I)$ which completes the proof. □

Let R_1 and R_2 be two rings, let δ_i be an expansion function of $\mathcal{I}(\mathcal{R}_i)$ for each $i\in\{1,2\}$ and $R=R_1\times R_2$. For a proper ideal $I_1\times I_2$, the function $\delta_\times$ defined by $\delta_\times(I_1\times I_2)=\delta_1(I_1)\times\delta_2(I_2)$ is an expansion function of $\mathcal{I(R)}$. The following result characterizes the 1-absorbing δ-primary ideals of the direct product of rings.

Theorem 2.22. *Let R_1 and R_2 be rings, $R=R_1\times R_2$ and let δ_i be an expansion function of $\mathcal{I}(\mathcal{R}_i)$ for $i=1,2$. Then the following statements are equivalent:*

(1) *I is an $(n,1)$-absorbing $\delta_\times$-primary ideal of R.*
(2) *I is a $\delta_\times$-primary ideal of R.*
(3) *Either $I=I_1\times R_2$, where I_1 is a δ_1-primary ideal of R_1 or $I=R_1\times I_2$, where I_2 is a δ_2-primary ideal of R_2 or $I=I_1\times I_2$, where I_1 and I_2 are proper ideals of R_1,R_2, respectively with $\delta_1(I_1)=R_1$ and $\delta_2(I_2)=R_2$.*

Proof. (1) $\Leftrightarrow$ (2). This follows from Theorem 2.5.
(2) $\Leftrightarrow$ (3) Let I be a $\delta_\times$-primary ideal of R. Hence I has the form $I=I_1\times I_2$ where I_1 and I_2 are ideals of R_1 and R_2 respectively. Without loss of generality, we may assume that $I=I_1\times R_2$ for some proper ideal I_1 of R_1. We show that I_1 is a δ_1-primary ideal of R_1. Deny. Then there are $a,b\in R_1$ such that $ab\in I_1, a\notin I_1$ and $b\notin\delta_1(I_1)$. Hence $(a,1)(b,1)\in I_1\times R_2$. Which implies that $(a,1)\in I_1\times R_2$ or $(b,1)\in\delta_\times(I_1\times R_2)$ and so $a\in I_1$ or $b\in\delta_1(I_1)$, which gives a contradiction. Now suppose that both I_1 and

I_2 are proper. As $(1,0)(0,1) \in I_1 \times I_2$ and $(1,0),(0,1) \notin I_1 \times I_2$, we have $(1,0),(0,1) \in \delta_\times(I_1 \times I_2) = \delta_1(I_1) \times \delta_2(I_2)$. Therefore $\delta_1(I_1) = R_1$ and $\delta_2(I_2) = R_2$. The converse is clear. □

In view of Theorem 2.22, we have the following result.

Theorem 2.23. *Let $R_1, R_2, \ldots, R_n$ be commutative rings with nonzero identity and $R = R_1 \times R_2 \times \cdots \times R_n$ where $n \geq 2$. Let δ_i be an expansion function of $\mathcal{I}(\mathcal{R}_i)$ for each $i = 1, 2, \ldots, n$. Then the following statements are equivalent.*

(1) I is an $(n,1)$-absorbing $\delta_\times$-primary ideal of R.

(2) $I = I_1 \times I_2 \times \cdots \times I_n$ and either for some $k \in \{1, 2, \ldots, n\}$ such that I_k is a δ_k-primary ideal of R_k and $I_j = R_j$ for each $j \in \{1, 2, \ldots, n\} \backslash \{k\}$ or I_{α_i}'s are proper ideals of R_{α_i} for $\{\alpha_1, \alpha_2, \ldots, \alpha_k\} \subseteq \{1, 2, \ldots, n\}$ and $|\{\alpha_1, \alpha_2, \ldots, \alpha_k\}| \geq 2$ with $\delta_{\alpha_i}(I_{\alpha_i}) = R_{\alpha_i}$, and $I_j = R_j$ for all $j \in \{1, 2, \ldots, n\} \backslash \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$.

Proof. It can be obtained by using mathematical induction on n. □

Let A be a ring and E an A-module. Then $A \ltimes E$, the *trivial (ring) extension of A by E*, is the ring whose additive structure is that of the external direct sum $A \oplus E$ and whose multiplication is defined by $(a,e)(b,f) := (ab, af + be)$ for all $a, b \in A$ and all $e, f \in E$. (This construction is also known by other terminology and other notation, such as the *idealization* $A(+)E$.) The basic properties of trivial ring extensions are summarized in the books [13], [12]. Trivial ring extensions have been studied or generalized extensively, often because of their usefulness in constructing new classes of examples of rings satisfying various properties (cf. [1, 2, 4, 8, 9, 14]). In addition, for an ideal I of A and a submodule F of E, $I \ltimes F$ is an ideal of $A \ltimes E$ if and only if $IE \subseteq F$. Moreover, for an expansion function δ of A, it is clear that $\delta_\ltimes$ defined as $\delta_\ltimes(I \ltimes F) = \delta(I) \ltimes E$ is an expansion function of $A \ltimes E$. Also as usual, if $c \in A$ then $(F : c) = \{e \in E \mid ce \in F\}$.

Theorem 2.24. *Let A be a ring, E an A-module and δ be an expansion function of $\mathcal{I}(A)$. Let I be an ideal of A and F a submodule of E such that $IE \subseteq F$. Then the following statement hold:*

(1) *If $I \ltimes F$ is an $(n,1)$-absorbing $\delta_\ltimes$-primary ideal of $A \ltimes E$, then I is an $(n,1)$-absorbing δ-primary ideal of A.*
(2) *Assume that $(F : c) = F$ for every $c \in A \setminus I$. Then $I \ltimes F$ is an $(n,1)$-absorbing $\delta_\ltimes$-primary ideal of $A \ltimes E$ if and only if I is an $(n,1)$-absorbing δ-primary ideal of A.*

Proof. (1) Assume that $I \ltimes F$ is an $(n,1)$-absorbing $\delta_\ltimes$-primary ideal of $A \ltimes E$ and let $x_1, ..., x_n, x_{n+1}$ be nonunit elements of A such that $x_1 ... x_n x_{n+1} \in I$. Thus $(x_1, 0)...(x_n, 0)(x_{n+1}, 0) = (x_1 ... x_n x_{n+1}, 0) \in I \ltimes F$ which implies that $(x_1, 0)...(x_n, 0) \in I \ltimes F$ or $(x_{n+1}, 0) \in \delta_\ltimes(I \ltimes F) = \delta(I) \ltimes E$. Therefore $x_1 ... x_n \in I$ or $x_{n+1} \in \delta(I)$ and so (1) holds.
(2) By (1), it suffices to prove the "if" assertion. Let $(x_1, e_1), ...(x_n, e_n), (x_{n+1}, e_{n+1})$

be nonunit elements of $A \ltimes E$ such that $(x_1, e_1)...(x_n, e_n)(x_{n+1}, e_{n+1}) \in I \ltimes F$. Clearly, $(x_1, e_1)...(x_n, e_n)(x_{n+1}, e_{n+1}) = (x_1...x_n, z)(x_{n+1}, e_{n+1}) = (x_1...x_n x_{n+1}, x_1...x_n e_{n+1} + x_{n+1}z)$ for some $z \in E$ and so $x_1...x_n x_{n+1} \in I$. Thus, $x_1...x_n \in I$ or $x_{n+1} \in \delta(I)$ since I is an $(n,1)$-absorbing δ-primary ideal of A. If $x_{n+1} \in \delta(I)$, then $(x_{n+1}, e_{n+1}) \in \delta(I) \ltimes E = \delta_{\ltimes}(I \ltimes F)$. Hence, we may assume that $x_{n+1} \notin \delta(I)$. Then $x_1...x_n \in I$. As $x_1...x_n e_{n+1} + x_{n+1}z \in F$ and $x_1...x_n e_{n+1} \in F$, we get that $x_{n+1}z \in F$. This implies that $z \in (F : x_{n+1}) = F$ and so $(x_1, e_1)...(x_n, e_n) \in I \ltimes F$. Therefore $I \ltimes F$ is an $(n,1)$-absorbing $\delta_{\ltimes}$-primary ideal of $A \ltimes E$. □

Corollary 2.25. *Let A be a ring, E an A-module and δ be an expansion function of $\mathcal{I}(A)$. Let I be a proper ideal of A. Then $I \ltimes E$ is an $(n,1)$-absorbing $\delta_{\ltimes}$-primary ideal of $A \ltimes E$ if and only if I is an $(n,1)$-absorbing δ-primary ideal of A.*

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Abdelhaq El Khalfi Fundamental and Applied Mathematics Laboratory., Faculty of Sciences Ain Chock, Hassan II University of Casablanca, Morocco.

E − mail : *abdelhaqelkhalfi@gmail.com*

Hinda Hamraoui Fundamental and Applied Mathematics Laboratory., Faculty of Sciences Ain Chock, Hassan II University of Casablanca, Morocco.

E − mail address : *hindahamraoui@yahoo.fr*

Hicham Laarabi Fundamental and Applied Mathematics Laboratory., Faculty of Sciences Ain Chock, Hassan II University of Casablanca, Morocco.

E − mail address : *hichamlaarabi036@gmail.com*